Abstract

The design of unbraced cold-formed steel beams must consider lateral-torsional buckling due to the low torsional stiffness associated with open cross-sections. The American Iron and Steel Institute incorporated design equations for the critical elastic lateral-torsional buckling stress in the North American Specification for the Design of Cold-Formed Steel Members. These equations are based on elastic theory for singly-symmetric and doubly-symmetric sections. However, the equation for point-symmetric sections is only a rough approximation. Furthermore, there are no provisions for lateral-torsional buckling of non-symmetric sections, or sections oriented to non-principal axes. This paper investigates and develops a general formulation of the lateral-torsional buckling equation to broadly cover all cold-formed steel cross-sections.

Introduction

Point-symmetric Zee sections are commonly used for structural members such as purlins, but the support directions do not typically align to the principal axes. The critical elastic lateral-torsional buckling stress is therefore more difficult to determine. The current AISI Specification provision for lateral-torsional buckling of point-symmetric sections is based on lateral-torsional buckling of a doubly-symmetric shape, with a reduction factor of 0.5 to roughly approximate its behavior. Numerical analysis has shown that this reduction factor can actually vary from 0.3 to 1.0 depending on section geometry.

It has also become more common in practice to use custom shapes as structural beams. This is often driven by application constraints, material optimization, and ease of material handling, among other factors. The current AISI Specification has no provisions for predicting the lateral-torsional buckling strength of non-symmetric sections, or beams where the support directions do not align with the principal axes.

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The lateral-torsional buckling equations used today for symmetrical shapes were originally investigated by Vlasov (1961) and Timoshenko (1961), and further studied by Peköz (1969). This paper expands on these developments to consider the more general case of any cold-formed steel cross-section at any orientation. Numerous symbols are used in this investigation which are defined at the end of this paper.

### Lateral-Torsional Buckling

An unbraced member subject to a sufficient bending moment may exhibit global buckling where the compression portion of the member translates laterally and rotates. Considering such a member oriented to its principal axes $u$ and $v$, with compression and bending applied to the ends of the member, the differential equations of equilibrium are given in Eq. 1, adapted from Vlasov (1961) and Peköz (1969).

\[
\begin{align*}
EL_v u''' + Pu'' + P(v_o - e_v)\phi'' &= 0 \\
EL_u v''' + Pv'' - P(u_o - e_u)\phi'' &= 0 \\
EC_w \phi''' - (GJ - 2\beta_v Pe_v - 2\beta_u Pe_u - Pr_o^2)\phi'' + P(v_o - e_v)u'' - P(u_o - e_u)v'' &= 0
\end{align*}
\] (1)

where the end moments are the product of the axial force $P$ and its biaxial eccentricities ($M_u = Pe_v, M_v = Pe_u$), and the following geometric properties of the cross-section are defined:

\[
\begin{align*}
\beta_v &= \frac{u_u}{2l_u} - v_o & \beta_u &= \frac{u_v}{2l_v} - u_o \\
U_u &= \int v^3dA + \int u^2vdA & U_v &= \int u^3dA + \int v^2udA \\
I_u &= \int v^2dA & I_v &= \int u^2dA
\end{align*}
\] (2)

To solve these differential equations, the displacements $u, v,$ and $\phi$ are assigned sinusoidal forms, which produce the following set of equations:

\[
\begin{align*}
(P_v - P)A_1 + (Pe_v - P\nu_o)A_3 &= 0 \\
(P_u - P)A_2 - (Pe_u - P\nu_u)A_3 &= 0 \\
(Pe_v - P\nu_o)A_1 - (Pe_u - P\nu_u)A_2 + [(P_t - P)r_o^2 - 2\beta_v Pe_v - 2\beta_u Pe_u]A_3 &= 0
\end{align*}
\] (5)

where

\[
\begin{align*}
P_u &= \pi^2 EI_u / L^2 & P_v &= \pi^2 EI_v / L^2 & P_t = \frac{1}{r_o^2}(GJ + \pi^2 EC_w / L^2)
\end{align*}
\] (6)
The solution to these simultaneous equations is obtained by equating the determinant of the coefficients on $A_1, A_2,$ and $A_3$ to zero:

$$\begin{vmatrix}
    P_v - P & 0 & Pe_v - P v_o \\
    0 & P_u - P & -Pe_u + P u_o \\
    Pe_v - P v_o & -Pe_u + P u_o & (P_t - P)r_o^2 - 2\beta_v Pe_v - 2\beta_u Pe_u
\end{vmatrix} = 0 \quad (7)$$

Expansion of this determinant gives the principal axis form of the flexural-torsional buckling equation for a member subjected to eccentric axial load:

$$\begin{align*}
(P_v - P)(P_u - P)[(P_t - P)r_o^2 - 2\beta_v Pe_v - 2\beta_u Pe_u] \\
-(P_u - P)(Pe_v - P v_o)^2 - (P_v - P)(Pe_u - P u_o)^2 &= 0 \quad (8)
\end{align*}$$

The development of a general form for non-principal axes would require a redevelopment of the differential equations of equilibrium to account for unsymmetric bending stress distributions in all three equations. This raises a number of complications which make it a difficult and undesirable approach.

This investigation pursues the problem by adapting the principal axis solution to a rotated coordinate system. Figure 1 shows an arbitrary cross-section with centroid $C$ and shear center $O$, oriented to orthogonal centroidal $x$ and $y$ axes which represent the directions of the supports. The principal $u$ and $v$ axes are oriented at an angle $\alpha$ measured counterclockwise from the $x$ and $y$ axes, respectively.

If the axial force $P$ is applied at point $E$ on the $y$ axis, the moment produced about the $x$ axis is $M_x = Pe_y$. The eccentricities associated with point $E$ relative to the principal axes are given by $e_u$ and $e_v$ as follows:

$$e_u = e_y \sin \alpha \quad e_v = e_y \cos \alpha \quad (9)$$

The application of a pure moment $M_x$ is achieved by increasing the eccentricity $e_v$ while decreasing the axial load $P$. Substituting the expressions in Eq. 9 into Eq. 8, and taking the limit as $e_y$ approaches infinity and $P$ approaches zero produces the following equation in terms of $M_x$:

$$P_u P_v P_t r_o^2 - 2\beta_v P_u P_v M_x \cos \alpha - 2\beta_u P_u P_v M_x \sin \alpha - P_u M_x^2 \cos^2 \alpha - P_v M_x^2 \sin^2 \alpha = 0 \quad (10)$$
This is rearranged into quadratic form as Eq. 11. It is then convenient to assign the nomenclature $P'_y$ and $\beta_y$ as defined in Eq. 12 to simplify the lateral-torsional buckling solution to Eq. 13.

$$\frac{P_u \cos^2 \alpha + P_v \sin^2 \alpha}{P_u P_v} M_x^2 + 2(\beta_y \cos \alpha + \beta_u \sin \alpha) M_x - P_r r_o^2 = 0 \quad (11)$$

$$P'_y = \frac{P_u P_v}{P_u \cos^2 \alpha + P_v \sin^2 \alpha} \quad \beta_y = \beta_v \cos \alpha + \beta_u \sin \alpha \quad (12)$$

$$M_x = P'_y \left[-\beta_y \pm \sqrt{\beta_y^2 + r_o^2 P_t / P'_y} \right] \quad (13)$$

Figure 1. Arbitrary cross-section oriented to x and y support directions

The same approach can be used for developing the lateral-torsional buckling moment about the y axis. If point E is placed on the x axis, producing moment $M_y = P e_x$, the following eccentricity relationships exist:

$$e_u = e_x \cos \alpha \quad e_v = -e_x \sin \alpha \quad (14)$$

Substituting the expressions in Eq. 14 into Eq. 8, and taking the limit as $e_x$ approaches infinity and $P$ approaches zero produces Eq. 15 in terms of $M_y$, with
the quadratic form shown as Eq. 16. Then assigning the nomenclature for \( P'_x \) and \( \beta_x \) as defined in Eq. 17 simplifies the \( M_y \) lateral-torsional buckling solution to Eq. 18.

\[
P_u P_v P_t r_0^2 + 2 \beta_v P_u P_v M_y \sin \alpha - 2 \beta_u P_u P_v M_y \cos \alpha - P_u M_y^2 \sin^2 \alpha - P_v M_y^2 \cos^2 \alpha = 0
\]  
(15)

\[
\frac{P_v \cos^2 \alpha + P_u \sin^2 \alpha}{P_u P_v} M_y^2 + 2 (\beta_u \cos \alpha - \beta_v \sin \alpha) M_x - P_t r_0^2 = 0
\]  
(16)

\[
P'_x = \frac{P_u P_v}{P_v \cos^2 \alpha + P_u \sin^2 \alpha} \quad \beta_x = \beta_u \cos \alpha - \beta_v \sin \alpha
\]  
(17)

\[
M_y = P'_x \left[ -\beta_x \pm \sqrt{\beta_x^2 + r_0^2 P_t / P'_x} \right]
\]  
(18)

**Axis Transformation**

The expressions for \( P' \) and \( \beta \) in equations 12 and 17 use principal axis properties \( I_u, I_v, \beta_u, \beta_v, U_u, \) and \( U_v \). Standard design procedures require section property calculations using the \( x \) and \( y \) axes which correspond to the member orientation. Numerical integration for both orientations requires additional effort, so the transformation of these properties between coordinate axes is beneficial.

The definitions for \( P'_y \) and \( P'_x \) in equations 12 and 17 can be factored as shown in Eq. 19. The principal axis moments of inertia must then be stated in terms of \( x \) and \( y \) axes.

\[
P'_y = \frac{\pi^2 E}{L^2} \frac{l_u l_v}{I_u \cos^2 \alpha + I_u \sin^2 \alpha}
\]

\[
P'_x = \frac{\pi^2 E}{L^2} \frac{l_u l_v}{I_v \cos^2 \alpha + I_u \sin^2 \alpha}
\]  
(19)

The location of each point in the cross-section is expressed in principal axis coordinates with the following relationships:

\[
u = x \cos \alpha + y \sin \alpha
\]

\[
v = y \cos \alpha - x \sin \alpha
\]  
(20)

Substituting Eq. 20 into the expressions for principal axis moments of inertia defined in Eq. 4 produces the following equations, where \( I_x \) and \( I_y \) are the moments of inertia about the \( x \) and \( y \) axes, and \( I_{xy} \) is the product of inertia.

\[
l_u = I_x \cos^2 \alpha + I_y \sin^2 \alpha - 2 I_{xy} \sin \alpha \cos \alpha
\]  
(21)

\[
l_v = I_y \cos^2 \alpha + I_x \sin^2 \alpha + 2 I_{xy} \sin \alpha \cos \alpha
\]  
(22)
From fundamental mechanics of materials, we recognize the following additional relationships derived using double-angle trigonometric identities and Mohr’s circle:

\[ \tan 2\alpha = \frac{-2I_{xy}}{I_x - I_y} \]  

(23)

\[ I_u, I_v = \frac{1}{2} (I_x + I_y) \pm \frac{1}{2} \sqrt{(I_x - I_y)^2 + 4I_{xy}^2} \]  

(24)

\[ I_u I_v = I_x I_y - I_{xy}^2 \]  

(25)

\[ I_x = I_u \cos^2 \alpha + I_v \sin^2 \alpha \quad I_y = I_v \cos^2 \alpha + I_u \sin^2 \alpha \]  

(26)

Substituting the relationships in Eq. 25 and Eq. 26 into Eq. 19 provides the definitions for \( P'_y \) and \( P'_x \) in terms of \( x \) and \( y \) section properties.

\[ P'_y = P_y \left( 1 - \frac{I_{xy}^2}{I_x I_y} \right) \quad P'_x = P_x \left( 1 - \frac{I_{xy}^2}{I_x I_y} \right) \]  

(27)

where

\[ P_y = \frac{\pi^2 E I_y}{l^2} \quad P_x = \frac{\pi^2 E I_x}{l^2} \]  

(28)

In a similar manner, the definitions for \( U_u \) and \( U_v \) can be stated in terms of \( x \) and \( y \) axis properties by substituting Eq. 20 into Eq. 3, which reduces to these straightforward transformations:

\[ U_u = U_x \cos \alpha - U_y \sin \alpha \quad U_v = U_x \cos \alpha + U_y \sin \alpha \]  

(29)

where

\[ U_x = \int y^3 dA + \int x^2 y dA \quad U_y = \int x^3 dA + \int y^2 x dA \]  

(30)

Then utilizing Eq. 20 for the shear center coordinates \((x_o, y_o)\) in Eq. 2, and substituting the results into the expressions for \( \beta \) in Eq. 12 and Eq. 17, provides the following relationships:

\[ \beta_y = \frac{U_u}{2I_u} \cos \alpha + \frac{U_v}{2I_v} \sin \alpha - y_o \quad \beta_x = \frac{U_v}{2I_v} \cos \alpha - \frac{U_u}{2I_u} \sin \alpha - x_o \]  

(31)

Further substitutions using Eq. 29 and the relationships in Eqs. 23 to 26 lead to these final forms in terms of \( x \) and \( y \) section properties:

\[ \beta_y = \frac{u_x I_y - u_y I_x}{2(I_x I_y - I_{xy}^2)} - y_o \quad \beta_x = \frac{u_y I_x - u_x I_y}{2(I_x I_y - I_{xy}^2)} - x_o \]  

(32)
Specific Cases

Principal Axes
If the support directions align with the principal axes, $I_{xy} = 0$. This simplifies the general solution such that $P_y$ and $P_x$ may be used in place of $P'_y$ and $P'_x$, and the properties $\beta_y$ and $\beta_x$ do not require transformation. The solution is reduced to the following:

$$M_x = P_y \left[ -\beta_y \pm \sqrt{\beta_y^2 + r_o^2 P_t / P_y} \right] \quad M_y = P_x \left[ -\beta_x \pm \sqrt{\beta_x^2 + r_o^2 P_t / P_x} \right]$$

$$\beta_y = \frac{u_x}{2l_x} - y_o \quad \beta_x = \frac{u_y}{2l_y} - x_o$$

Point-Symmetric
For point-symmetric sections, the shear center coincides with the centroid (i.e., $x_o = 0$, $y_o = 0$). Furthermore, the properties $U_x$ and $U_y$ are equal to zero, thus $\beta_y$ and $\beta_x$ are also zero. The lateral-torsional buckling equations take the following simpler form:

$$M_x = \pm r_o \sqrt{P_y P_t} \quad M_y = \pm r_o \sqrt{P_x P_t}$$

Symmetric About X Axis
For any section symmetric about the $x$ axis, including doubly-symmetric sections, the properties $I_{xy}$, $U_x$, $y_o$, and $\beta_y$ are all zero. Therefore the lateral-torsional buckling equation for bending about the $x$ axis is simply:

$$M_x = \pm r_o \sqrt{P_y P_t}$$

Symmetric About Y Axis
For any section symmetric about the $y$ axis, including doubly-symmetric sections, the properties $I_{xy}$, $U_y$, $x_o$, and $\beta_x$ are all zero. Therefore the lateral-torsional buckling equation for bending about the $y$ axis is simply:

$$M_y = \pm r_o \sqrt{P_x P_t}$$

Fully Braced in X Direction
If a member is fully braced in the $x$ direction, $P'_y$ approaches infinity and $1/P'_y$ becomes zero. The $M_{x}^2$ term in Eq. 11 drops out, thus reducing the solution to Eq. 38. There is only one root to the equation, so the sign of $\beta_x$ dictates the sign
of the torsional buckling moment. If $\beta_y$ is very small, the member is not subject to torsional buckling.

$$M_x = r_o^2 P_t / 2\beta_y$$  \hfill (38)

**Fully Braced in Y Direction**

If a member is fully braced in the $y$ direction, $P'_x$ approaches infinity and $1/P'_x$ becomes zero. The $M_y^2$ term in Eq. 16 drops out, thus reducing the solution to Eq. 39. There is only one root to the equation, so the sign of $\beta_t$ dictates the sign of the torsional buckling moment. If $\beta_t$ is very small, the member is not subject to torsional buckling.

$$M_y = r_o^2 P_t / 2\beta_t$$  \hfill (39)

**Stress Representation**

The above lateral-torsional buckling moment equations were developed using axial compressive forces. These can be restated using compressive stresses, where axial stress $\sigma = P/A$.

<table>
<thead>
<tr>
<th>$M_x$</th>
<th>$M_y$</th>
</tr>
</thead>
</table>
| $M_x = A\sigma_{ey}[-\beta_y \pm \sqrt{\beta_y^2 + r_o^2 \sigma_t / \sigma_{ey}}]$ | $M_y = A\sigma_{ex}[-\beta_x \pm \sqrt{\beta_x^2 + r_o^2 \sigma_t / \sigma_{ex}}]$ | \hfill (40)

$$\sigma_{ey}' = \sigma_{ey} \left( 1 - \frac{l_x^2}{l_y l_y} \right) \quad \sigma_{ex}' = \sigma_{ex} \left( 1 - \frac{l_x^2}{l_y l_y} \right)$$  \hfill (41)

$$\sigma_{ey} = \frac{\pi^2 E}{(L/r_y)^2} \quad \sigma_{ex} = \frac{\pi^2 E}{(L/r_x)^2}$$  \hfill (42)

$$\sigma_t = \frac{1}{Ar_o^2} \left( GJ + \pi^2 EC_w / L^2 \right)$$  \hfill (43)

**Principal Axes**

<table>
<thead>
<tr>
<th>$M_x$</th>
<th>$M_y$</th>
</tr>
</thead>
</table>
| $M_x = A\sigma_{ey}[-\beta_y \pm \sqrt{\beta_y^2 + r_o^2 \sigma_t / \sigma_{ey}}]$ | $M_y = A\sigma_{ex}[-\beta_x \pm \sqrt{\beta_x^2 + r_o^2 \sigma_t / \sigma_{ex}}]$ | \hfill (44)

**Point-Symmetric**

$$M_x = \pm Ar_o \sqrt{\sigma_{ey}' \sigma_t} \quad M_y = \pm Ar_o \sqrt{\sigma_{ex}' \sigma_t}$$  \hfill (45)
Symmetric About X Axis

\[ M_x = \pm Ar_o \sqrt{\sigma_{ey} \sigma_t} \]  \hspace{1cm} (46)

Symmetric About Y Axis

\[ M_y = \pm Ar_o \sqrt{\sigma_{ex} \sigma_t} \]  \hspace{1cm} (47)

Fully Braced in X Direction

\[ M_x = Ar_o^2 \sigma_t / 2\beta_y \]  \hspace{1cm} (48)

Fully Braced in Y Direction

\[ M_y = Ar_o^2 \sigma_t / 2\beta_x \]  \hspace{1cm} (49)

Illustrative Example

Given the eave strut section shown in Figure 2 with the section properties provided in Table 1, determine the positive and negative lateral-torsional buckling moments about the x and y axes for various unbraced lengths.

Figure 2. Eave Strut 8x5x3x14ga
### Table 1: Section Properties for Eave Strut 8x5x3x14ga

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.162 in²</td>
<td>Iₓᵧ</td>
<td>−1.754 in⁴</td>
</tr>
<tr>
<td>Iₓ</td>
<td>12.317 in⁴</td>
<td>Iᵧ</td>
<td>2.873 in⁴</td>
</tr>
<tr>
<td>Uₓ</td>
<td>5.035 in⁵</td>
<td>Uᵧ</td>
<td>9.637 in⁵</td>
</tr>
<tr>
<td>rₓ</td>
<td>3.256 in</td>
<td>rᵧ</td>
<td>1.572 in</td>
</tr>
<tr>
<td>xₒ</td>
<td>−2.771 in</td>
<td>yₒ</td>
<td>−1.574 in</td>
</tr>
<tr>
<td>Iₒ</td>
<td>26.991 in⁴</td>
<td>rₒ</td>
<td>4.820 in</td>
</tr>
<tr>
<td>J</td>
<td>0.001844 in⁴</td>
<td>Cₛ</td>
<td>22.89 in⁶</td>
</tr>
</tbody>
</table>

Calculate βᵧ and βₓ using Eq. 32

\[
\beta_y = \frac{(5.035)(2.873) - (9.637)(-1.754)}{2[(12.317)(2.873) - (-1.754)^2]} - (-1.574) = 2.059 \text{ in}
\]

\[
\beta_x = \frac{(9.637)(12.317) - (5.035)(-1.754)}{2[(12.317)(2.873) - (-1.754)^2]} - (-2.771) = 4.744 \text{ in}
\]

Calculate positive \( Mₓ \) for \( L = 300 \text{ in} \)

\[
\sigma_t = \frac{1}{1.162(4.820)^2} [11300(0.001844) + \pi^2 29500(22.89)/300^2] = 3.515 ksi
\]

\[
\sigma_{t'y} = \frac{\pi^2 29500}{(300/1.572)^2} \left[ 1 - \frac{(-1.754)^2}{(12.317)(2.873)} \right] = 7.299 ksi
\]

\[
M_x = 1.162(7.299)\left[ -2.059 + \sqrt{2.059^2 + 4.820^2(3.515/7.299)} \right] = 15.85 \text{ k-in}
\]

Calculate positive \( Mᵧ \) for \( L = 300 \text{ in} \)

\[
\sigma_t = \frac{1}{1.162(4.820)^2} [11300(0.001844) + \pi^2 29500(22.89)/300^2] = 3.515 ksi
\]

\[
\sigma_{t'x} = \frac{\pi^2 29500}{(300/3.256)^2} \left[ 1 - \frac{(-1.754)^2}{(12.317)(2.873)} \right] = 31.31 ksi
\]

\[
M_y = 1.162(31.31)\left[ -4.744 + \sqrt{4.744^2 + 4.820^2(3.515/31.31)} \right] = 9.73 \text{ k-in}
\]
Calculate negative $M_x$ for $L = 480$ in

$$\sigma_t = \frac{1}{1.162(4.820)^2} [11300(0.001844) + \pi^2 29500(22.89)/480^2] = 1.843 \text{ ksi}$$

$$\sigma_e' = \frac{\pi^2 29500}{(480/1.572)^2} \left[ 1 - \frac{(-1.754)^2}{(12.317)(2.873)} \right] = 2.851 \text{ ksi}$$

$$M_x = 1.162(2.851)[-2.059 - \sqrt{2.059^2 + 4.820^2(1.843/2.851)}] = -21.36 \text{ k-in}$$

Calculate negative $M_y$ for $L = 960$ in

$$\sigma_t = \frac{1}{1.162(4.820)^2} [11300(0.001844) + \pi^2 29500(22.89)/960^2] = 1.040 \text{ ksi}$$

$$\sigma_e' = \frac{\pi^2 29500}{(960/3.256)^2} \left[ 1 - \frac{(-1.754)^2}{(12.317)(2.873)} \right] = 3.058 \text{ ksi}$$

$$M_y = 1.162(3.058)[-4.744 - \sqrt{4.744^2 + 4.820^2(1.040/3.058)}] = -36.45 \text{ k-in}$$

Finite strip analyses for these cases produced the following results, which are within 0.5% of the calculated values: $M_x = 15.85$ k-in, $M_y = 9.71$ k-in, $M_x = -21.34$ k-in, $M_y = -36.29$ k-in.

Impact on Design

For singly-symmetric and doubly-symmetric sections, the AISI (2016) provisions are equivalent to Eqs. 44, 46, and 47. For point-symmetric sections, the AISI provisions apply a reduction factor of 0.5 to Eq. 46. However, this reduction factor should depend on the section geometry as reflected in Eq. 45. The ratio of Eq. 45 to Eq. 46 quantifies the reduction factor as $\sqrt{1 - I_{xy}^2/I_x I_y}$.

The AISI Design Manual (2013) contains several tables and charts for ordinary Zee sections. Table 2 below provides a comparison of the elastic buckling stress calculations for these sections, where one thickness was chosen to represent each size. The 0.5 reduction factor used in the current AISI provisions is very conservative for these sections, averaging 27% below the theoretical elastic buckling stress. The finite strip method (FSM) provided elastic buckling stresses which essentially match the theoretical values.
Table 2: Lateral-torsional buckling stress for various Zee shapes

<table>
<thead>
<tr>
<th>Section (L = 180 in)</th>
<th>F_{cre} (ksi)</th>
<th>F_{cre AISI} (ksi)</th>
<th>F_{cre FSM} (ksi)</th>
<th>F_{cre AISI} / F_{cre}</th>
<th>F_{cre FSM} / F_{cre}</th>
</tr>
</thead>
<tbody>
<tr>
<td>12ZS3.25x105</td>
<td>21.43</td>
<td>15.24</td>
<td>21.23</td>
<td>0.711</td>
<td>0.991</td>
</tr>
<tr>
<td>12ZS2.75x105</td>
<td>16.52</td>
<td>11.50</td>
<td>16.42</td>
<td>0.696</td>
<td>0.994</td>
</tr>
<tr>
<td>12ZS2.25x105</td>
<td>12.20</td>
<td>8.29</td>
<td>12.14</td>
<td>0.679</td>
<td>0.996</td>
</tr>
<tr>
<td>10ZS3.25x105</td>
<td>22.09</td>
<td>16.15</td>
<td>21.95</td>
<td>0.731</td>
<td>0.994</td>
</tr>
<tr>
<td>10ZS2.75x105</td>
<td>17.22</td>
<td>12.32</td>
<td>17.14</td>
<td>0.716</td>
<td>0.995</td>
</tr>
<tr>
<td>10ZS2.25x105</td>
<td>12.90</td>
<td>9.01</td>
<td>12.86</td>
<td>0.698</td>
<td>0.997</td>
</tr>
<tr>
<td>9ZS2.25x105</td>
<td>13.33</td>
<td>9.46</td>
<td>13.29</td>
<td>0.709</td>
<td>0.997</td>
</tr>
<tr>
<td>8ZS3.25x105</td>
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<td>17.32</td>
<td>22.83</td>
<td>0.755</td>
<td>0.994</td>
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<td>13.41</td>
<td>18.08</td>
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<td>0.996</td>
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<td>10.66</td>
<td>14.52</td>
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<tr>
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<td>9.01</td>
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</tbody>
</table>

Average 0.727 0.997
Std Dev 0.027 0.003

The proportions of these Zee sections are similar, so the level of conservatism (22% to 32%) is fairly consistent. However, the product of inertia \( I_{xy} \) is sensitive to the web angle of a Zee section.

Figure 3 illustrates how changes to the web angle for the sections in Table 2 impact the ratio of the AISI buckling stress to the theoretical buckling stress, which varies by ±50%. For extreme cases where \( I_{xy}^2 \) approaches \( I_xI_y \), the theoretical buckling stress approaches zero and the AISI provisions become very unconservative.

The AISI Specification provides an alternate, simpler equation, which is also plotted in Figure 3. For Zee sections with 90° webs, this equation provides acceptable, conservative results. For other web angles, the alternate equation is either very conservative or very unconservative.
For angles and any other sections not oriented to the principal axes, there are no AISI provisions for lateral-torsional buckling, although numerical analyses such as the finite strip method may be applied as a rational analysis.

Conclusions

A general lateral-torsional buckling equation has been developed which is applicable to any cold-formed steel shape. Two factors in this equation were defined using principal axis properties of the cross-section, but axis transformations developed herein permit calculation of these factors using $x$ and $y$ axis section properties which correspond to the member orientation.

Buckling stress predictions were compared to numerical solutions for a variety of sections and lengths. The finite strip method provided very good agreement. Cases with large slenderness had extremely close results, whereas slight deviations were observed as slenderness decreased.

This development fulfills a specific need in the industry to accurately predict lateral-torsional buckling strength for point-symmetric and non-symmetric shapes. The current AISI provisions for point-symmetric sections were shown to be overly conservative for common Zee shapes. For some less common point-symmetric sections, the AISI provisions could be very unconservative. It is
therefore recommended that the AISI provisions be modified to use this elastic buckling equation.

Currently AISI has no provisions for lateral-torsional buckling of non-symmetric shapes. The inclusion of this general buckling equation will benefit the engineer so that more complex rational methods such as finite strip analysis are not required.

**Notation**

- \( A \) Area of cross-section
- \( C_w \) Torsional warping constant
- \( E \) Modulus of elasticity
- \( e_u, e_v \) Eccentricity of axial load relative to \( u \) and \( v \) axes
- \( e_x, e_y \) Eccentricity of axial load relative to \( x \) and \( y \) axes
- \( G \) Shear modulus of elasticity
- \( J \) Saint-Venant torsion constant
- \( I_u, I_v \) Moment of inertia about principal \( u \) and \( v \) axes
- \( I_x, I_y \) Moment of inertia about \( x \) and \( y \) axes
- \( I_{xy} \) Product of inertia about \( x \) and \( y \) axes
- \( L \) Beam length
- \( M_x, M_y \) Critical elastic buckling moment about \( x \) and \( y \) axes
- \( P \) Critical elastic buckling axial load
- \( P_u, P_v \) Critical axial load for elastic buckling about principal \( u \) and \( v \) axes
- \( P_x, P_y, P_t \) Critical axial load for elastic buckling about \( x \) axis, \( y \) axis, and torsion
- \( P'_x, P'_y \) Adjusted axial load for elastic buckling about non-principal \( x \) and \( y \) axes
- \( r_o \) Polar radius of gyration about shear center
- \( r_x, r_y \) Radius of gyration about \( x \) and \( y \) axes
- \( U_u, U_v \) Geometric properties of cross-section as defined in Eq. 3
- \( U_x, U_y \) Geometric properties of cross-section as defined in Eq. 30
- \( u, v \) Principal coordinate axes of cross-section
- \( u, v, \phi \) Buckling displacements in the \( u \) and \( v \) directions, and angle of twist
- \( u'', v'', \phi'' \) Second derivative of buckling displacements with respect to longitudinal axis
- \( u''', v''', \phi''' \) Fourth derivative of buckling displacements with respect to longitudinal axis
- \( u_o, v_o \) Principal axis coordinates of shear center relative to centroid
$x, y$  Coordinate axes of cross-section corresponding to support directions

$x_0, y_0$  Coordinates of shear center relative to centroid

$\alpha$  Angle of $u$ principal axis measured counter-clockwise from $x$ axis

$\beta_u, \beta_v$  Geometric properties of cross-section as defined in Eq. 2

$\beta_x, \beta_y$  Geometric properties of cross-section as defined in Eq. 32

$\sigma_{ex}, \sigma_{ey}, \sigma_t$  Critical axial stress for elastic buckling about $x$ axis, $y$ axis, and torsion

$\sigma'_{ex}, \sigma'_{ey}$  Adjusted axial stress for elastic buckling about non-principal $x$ and $y$ axes

References


